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ABSTRACT

This paper describes a research and development project in teaching designed to examine whether and how it might be possible to bring the meaning of knowing mathematics in school closer to what it means to know mathematics within the discipline by deliberately altering the roles and responsibilities of teacher and students in classroom discourse. The project was carried out as a regular feature of lessons in fifth-grade mathematics in a public school. The teacher modeled a form of social interaction appropriate to making mathematical arguments in response to students' conjectures. Students were taught to examine the hypotheses about mathematical structures that underlie their solutions to problems. The outcomes of this teaching are described in terms of the abilities and propensities students develop to create mathematical arguments in a particular episode of classroom interaction. (Author)

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THE TEACHER'S ROLE IN REINVENTING THE MEANING OF MATHEMATICAL KNOWING IN THE CLASSROOM

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Abstract

This paper describes a research and development project in teaching designed to examine whether and how it might be possible to bring the meaning of knowing mathematics in school closer to what it means to know mathematics within the discipline by deliberately altering the roles and responsibilities of teacher and students in classroom discourse. The project was carried out as a regular feature of lessons in fifth-grade mathematics in a public school. The teacher modeled a form of social interaction appropriate to making mathematical arguments in response to students' conjectures. Students were taught to examine the hypotheses about mathematical structures that underlie their solutions to problems. The outcomes of this teaching are described in terms of the abilities and propensities students develop to create mathematical arguments in a particular episode of classroom interaction.

THE TEACHER'S ROLE IN REINVENTING THE MEANING OF MATHEMATICAL KNOWING IN THE CLASSROOM*

Magdalene Lampert**

A Mathematical Preface

In the midst of an argument among his students about a theorem in geometry, the teacher in Lakatos's Proofs and Refutations finds it appropriate to announce: "I respect conscious guessing, because it comes from the best human qualities: courage and modesty" (1976, p. 30). Why does this teacher of mathematics think it appropriate to encourage conscious guessing and to celebrate the human virtues of courage and modesty? The answer to this question is to be found in Lakatos's analysis of what it means to know mathematics and his ideas about how new knowledge develops in the discipline.

In Proofs and Refutations, Lakatos portrays historical debates within mathematics about what the "proof" of a theorem represents by constructing a conversation among a group of students: fictional characters who voice the disagreements among mathematicians through the last several centuries, often using the mathematicians' own words. Lakatos's argument, which comes through in the person of the teacher, is that mathematics develops as a process of "conscious guessing" about relationships among quantities and shapes, with proof following a "zig-zag" path starting from conjectures and moving to the

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examination of premises through the use of counterexamples or "refutations." This activity of doing mathematics is different from what is recorded once it is done: "Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end product" (Lakatos, 1976, p. 42). The product of mathematical activity might be justified with a deductive proof, but the product does not represent the process of coming to know. Nor is knowing final or certain, even with a proof, for the assumptions on which the proof is based--which mathematicians call axioms--continue to be open to reexamination in the mathematical community of discourse.[1] It is this vulnerability to reexamination that allows mathematics to grow and develop.

Mathematics has grown and changed over time, in Lakatos's view, not because the conclusions that are derived from axioms are the result of faulty logic, but because the axioms and definitions from which the logical argument begins are themselves open to revision as they are examined in the community of discourse. The need for revisions does not become obvious, however, until one engages in the process of proof and discovers the shortcomings of one's assumptions. The insufficiencies of the original assumptions come to be recognized as one tries to pursue the logical consequences rather than before the fact: refutations of the conclusions, often in the form of counter-examples, suggest revisions to the assumptions. Lakatos demonstrated that this zig-zag between revising conclusions and revising assumptions in the process of coming to know occurred both in the work of individual mathematicians as they exposed their work to their colleagues and over time and as conclusions that had been unquestioned in the past were reconsidered.

From the standpoint of the person doing mathematics, making a conjecture (or what Lakatos calls a "conscious guess") is taking a risk; it requires the

admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. While possibly garnering recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability. Courage and modesty are appropriate to participation in mathematical activity because truth remains tentative, even as the proof of a conjecture evolves.

Polya (1954) also thought courage and modesty to be essential to the activity of acquiring mathematical knowledge. He asserted that the doer of mathematics must assume "the inductive attitude" and be willing to question both observations and generalizations, playing them off of one another in a form similar to what Lakatos called the zig-zag path from conjecture to proof and back to axioms. Polya asserted:

> In our personal life we often cling to illusions. That is we do not dare to examine certain beliefs which could be easily contradicted by experience, because we are afraid of upsetting the emotional balance. There may be circumstances in which it is not unwise to cling to illusions, but [in doing mathematics] . . . we need to adopt the inductive attitude [which] requires a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations. It requires saying "maybe" and "perhaps" in a thousand different shades. It requires many other things, especially the following three:
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> | INTELLECTUAL COURAGE: | we should be ready to revise any one of our beliefs. |
> | INTELLECTUAL HONESTY: | we should change a belief when there is a good reason to change it. . . . |
> | WISE RESTRAINT: | we should not change a belief wantonly, without some good reason, without serious examination. (pp. 7-8) |

Polya called these the "moral qualities" required to do mathematics. He recognized that examining one's assumptions is an emotionally risky matter, but like Lakatos, he claimed that it was essential to doing good mathematics.

Coming to Know Mathematics in School

These views of what doing and knowing mathematics entail are radically different from those that are assumed in popular culture, and in most classrooms.[2] Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly (Ball, 1988; Schoenfeld, 1985b; Stodolsky, 1985). These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing.

Comparing School Mathematics With Knowing in the Discipline

The issue of intellectual authority is central to this comparison between how mathematics is known in school and how it is known in the discipline. In the classroom, the teacher and the textbook are the authorities and mathematics is not a subject to be created or explored. In school, the truth is given in the teacher's explanations and the answer book; there is no zig-zag between conjectures and arguments for their validity, and one could hardly imagine hearing the words "maybe" or "perhaps" in a lesson. Knowing mathematics in school therefore comes to mean having a set of unexamined beliefs, whereas Lakatos and Polya suggest that the knower of mathematics needs to be able to stand back from his or her own knowledge, evaluate its antecedent assumptions, argue about the foundations of its legitimacy, and be willing to have others do the same.

Although the teacher tells students whether their answers are right or wrong, it seems that few teachers engage students in a public analysis of the

assumptions that they make to get their answers. Even when teachers give an explanation rather than simply stating a rule to be followed, they do not invite students to examine the mathematical assumptions behind the explanation, and it is unlikely that they do so themselves (Ball, in press). In conventional mathematics lessons, students believe that the teacher knows which answers are right, and teachers believe that the paths to these answers can be found in rules in books; examining these beliefs, in Polya's words, "can upset the emotional balance" in both teachers and students (Cooney, 1987). That teachers and students think this way about mathematical knowledge and how it is acquired is both a cause and a logical consequence of the ways in which knowledge is regarded in school mathematics lessons.

Must There Be This Disjunction?

This paper is a description of a research and development project in mathematics teaching designed to explore whether and how it might be possible to produce lessons in which public school students would exhibit--in the classroom--the qualities of mind and morality that Lakatos and Polya associate with doing mathematics.[3] My role in this project has been to develop and implement new forms of teacher-student interaction as well as to experiment with new forms of content as a teacher of fifth-grade mathematics. The findings of this research and development project will be presented here in terms of a story about learning about knowing mathematics in the social setting of the classroom. The end of the story will be told first by describing the activity of a class of fifth graders who seem to have learned to do mathematics together in a way that is consonant with Lakatos's and Polya's assertions about what doing and knowing mathematics entails.

The evidence for their learning will be presented in the form of an exhibit of what they are able and inclined to do with their teacher and with

one another in the social context of a lesson. The students are courageous and modest in making and evaluating their own assertions and those of others, and in arguing about what is mathematically true; they move around in their thinking from observations to generalizations and back to observations to refute their own ideas and those of their classmates. While they are learning mathematics, these students are also learning, tacitly if not explicitly, to place mathematics appropriately in the lexicon of ways of knowing. Their activity suggests that they are operating with quite a different set of beliefs about what doing mathematics means than those held by other fifth graders in similar school settings (Stodolsky, 1988). In acting on what they believe about the relationship between the knower and what is known, they put themselves in the position of authors of ideas and arguments; in their talk about mathematics, reasoning and mathematical argument--not the teacher or the textbook--are the primary source of an idea's legitimacy.

The research reported here is intended to be transformative (Silver, in press): It is not a study of currently conventional practices in mathematics teaching. Instead, it is intended to portray and analyze a type of practice that might be desirable. As a vision of what is possible in a public school classroom, it represents the sort of dramatic changes in the nature of precollege mathematics instruction that are called for in such documents as Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 1987). At every level of schooling, and for all students, the Standards volume recommends that mathematics students should be making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions, and discussing and questioning their own thinking and the thinking of others. These activities do not fit within the tasks that currently define mathematics lessons. Moreover, they

require both teachers and students to think differently about the nature of mathematical knowledge. Very little research has examined either the intellectually generative sort of mathematical activities espoused by NCTM or the role that the classroom culture plays in the social construction of a view of mathematical knowledge that would support such activities; studies of this sort are needed if we are to understand what it will take to transform standards into practice (NCTM, 1988).

How Does Learning What It Means to Know Mathematics Come About?

In addition to illustrating what the students have learned about how to participate in the doing and learning of mathematics, the case study of teaching and learning that is the focus of this paper will also show how the teacher acts to create and maintain the culture in which such student activity can occur. This particular case was chosen to represent the mathematics lessons I have designed and taught because it illustrates several patterns of teacher and student interaction that can be shown to be common to almost every lesson over the entire school year. Research suggests that such patterns can be initiated by the teacher to build a "participation structure" that redefines the roles and responsibilities of both teacher and students in relation to learning and knowing (Au & Jordan, 1980; Au & Mason, 1981). The notion of a classroom participation structure is taken from the sociolinguistic work of Florio (1978) and Erickson and Shultz (1981). They define a participation structure to be the allocation of interactional rights and obligations among participants in a social event; it represents the consensual expectations of the participants about what they are supposed to be doing together, their relative rights and duties in accomplishing tasks, and the range of behaviors appropriate within the event.[4] Teachers and students form communities of

discourse that come to agree on working definitions of what counts as knowledge and the processes whereby knowledge is assumed to be acquired (Cazden, 1988). Thus the classroom comes to parallel the mathematical community more closely, as truth is determined by logical argument among scholars (Balachef, 1987; Davis & Hersch, 1981).

Shared Meaning

In the classroom, words like "know," "think," "revise," "explain," "problem," and "answer" come to have meaning by being associated with particular kinds of activities. Who is responsible for doing the activities associated with these words gets determined in interaction between the teacher and the students. In conventional lessons the participants agree that it is the teacher's responsibility to explain and the students' responsibility to give answers. Problems are questions, and finding the correct answers to those questions is an indication of knowing mathematics. The teacher is responsible for knowing whether the answers a student gives are correct and to be asked by the teacher to revise means "you've done it wrong" and "you've got to do it over" until more of the answers are correct. When the teacher asks a students to think it is often an admonition to be quiet; thinking is not considered to be a process that students--or teacher and students--engage in together.

What Is the Teacher's Role?

Teaching is not only about teaching what is conventionally called "content." It is also teaching students what a lesson is and how to participate in it (Florio, 1978; Jackson, 1968; Mehan, 1979). From the activities the teacher sets for them, students learn what counts as knowledge and what kind of activities constitute legitimate academic tasks (Cazden,

1988; Doyle, 1985, 1986; Leinhardt & Putnam, 1987; Lemke, 1982). Face-to-face interaction between students and their teacher follows context-specific rules, and cues within these contexts signal how what anyone says is to be understood in relation to the task everyone is assembled to accomplish (Cazden, 1988; Mehan, 1979). The teacher has more power over how acts and utterances get interpreted, being in a position of social and intellectual authority, but these interpretations are finally the result of negotiation with students about how activity is to be regarded.

In order to challenge conventional assumptions about what it means to know mathematics, then, teachers and students need to do different sorts of activities together, with different kinds of roles and responsibilities. In my interactions with students, I associated mathematically appropriate activities with words like "know," "think," "revise," "explain," "problem," and "answer" to initiate a redefinition of these activities, and I demonstrated to students what new roles and responsibilities--for them and for me--the new definitions entailed.

When the Problem Is Not the Question and the Solution Is Not the Answer: Inventing New Forms of Teacher-Student Interaction

How does a teacher[5] go about redefining the meaning of "knowing mathematics?" In the lessons I designed and enacted, I portrayed what I wanted students to learn about mathematical knowing both in how I constructed my own role and in what I expected of the class. I gave them <u>problems</u> to do, but <u>I</u> did not <u>explain</u> how to get the answers, and the questions I expected them to answer went beyond simply determining whether they could get the <u>solutions</u>. I also expected them to answer questions about mathematical assumptions and the legitimacy of their strategies. <u>Answers</u> to problems were given by students, but I did not interpret them to be the primary indication of whether they <u>knew</u>

mathematics. In this interaction, the words "knowing," "revising," "thinking," "explaining," "problem," and "answer" took on new meanings.

Students' Knowing as Conjecture and Proof

In setting the topical agenda in any particular lesson--that is, in determining what the mathematical discourse was going to be about--my responsibility was to choose a problem or problems for discussion and the students' responsibility was to express their interests, questions, and understandings within the domain of the problem. As it was realized during a discussion the agenda evolved from a negotiation between the problems I set and the mathematics that students brought to working on those problems (cf. Steffe, 1988). The problems communicated predictable boundaries for the class discussion, enabling students to know what they are supposed to be doing and thinking about during the class period (cf. Leinhardt & Putnam, 1987).

At the beginning of a unit, when we were switching to a new topic, the problem we started with was chosen for its potential to expose a wide range of students' thinking about a bit of mathematics, to make explicit and public what they could do, and how they understand. Later problems were chosen based on an assessment of the results of the first and subsequent discussions of a topic, moving the agenda along into novel but related mathematical territory. The most important criterion in picking a problem was that it be the sort of problem that would have the capacity to engage all of the students in the class in making and testing mathematical hypotheses. These hypotheses are embedded in the answers students give to the problem, and so comparing answers engaged the class in a discussion of the relative mathematical merits of various hypotheses, setting the stage for the kind of zig-zag between inductive observation and deductive generalization that Lakatos and Polya see as characteristic of mathematical activity.

For example, in the class described below, students asserted various hypotheses about how to figure out the last digit in 5^4, 6^4, and 7^4 without multiplying.[6] In order to push them to speculate on whether their hypotheses would hold in a larger domain, I then asked them to think about what the last digit would be in 7^5. Two competing hypotheses about how exponents work were revealed in their assertions about the last digit in 7^5. One student's initial hypothesis was that the fifth power would be obtained by squaring the fourth power. Other students hypothesized that the fifth power would be obtained by multiplying the fourth power by the base number (or first power). In arguing about whether the last digit in 7^5 was 1 or 7, the students were arguing about which of these hypotheses was an appropriate "law" to use in their further work with exponents.

Problems like these are what Kilpatrick (1987), following Frederickson, calls "structured problems requiring productive thinking" (p. 134). Such problems imply criteria for testing the correctness of the solution, but they are not solved by the simple application of a known algorithm. This allows for multiple routes to a solution, and puts the solver in the position of devising all or part of the solution procedure. The students' responsibility is figuring out how to solve the problem as well as finding the solution. It is the strategies used for figuring out, rather than the answers, that are the site of the mathematical argument, and it is these strategies that reveal the assumptions a student is making about how mathematics works.

The intellectual <u>problem</u> for the students is developing a mathematically legitimate strategy for finding the answer to a question posed by the teacher. The content of the lesson is the arguments that support or reject solution strategies rather than the finding of answers. Students' strategies yield answers to teachers' questions, but the <u>solution</u> is more than the

answer, just as the problem is more than the question. Generating a strategy and arguing for its legitimacy indicates what the student knows about mathematics. These are the activities of significance in the verbal and nonverbal communication between teacher and students. It is in this sense that "the problem is not the question and the answer is not the solution." This approach to teacher and students talking about the problem and its solutions is intended to communicate that what is important is developing and defending strategies, making hypotheses, or what Lakatos calls "conscious guessing," and rising to the challenge of articulating and defending the assumptions that led up to your guess.

How Does Mathematical Argument in the Classroom Work to Express Mathematical Knowledge?

As students volunteered their solutions to a given problem, I wrote them all up on the board for consideration, and I put a question mark next to all of them. Often I put students' names next to their answers, as a help for everyone to remember who they are to address if they have something to say about that answer. The names also are meant to indicate that the answers still belong to the persons who figured them out, even though they have been given to the teacher. When someone asserted that one of the answers should be eliminated because it was incorrect, it was considered fair game for the teacher to ask anyone in the class to explain why.

Once the list of students' solutions was up on the board, they were open for discussion and revision. Students often began by explaining why they gave the answer that they did. If they wanted to disagree with an answer that was up on the board, the language that I have taught them to use is, "I want to question so-and-so's hypothesis." (Until the group arrived at a mutually agreed-upon proof that one or more of the answers must be correct, all answers

were considered to be "hypotheses.") I always asked them to give reasons why they questioned the hypothesis so that their challenge takes the form of a logical refutation rather than a judgment. The person who gave the answer was free to respond or not with a revision. In order to communicate the idea that I thought every answer was (or should be) arrived at by a process of reasoning that makes sense to the person who volunteered it, I asked the class, "Can anyone explain what they thought so-and-so was thinking?" and "Why would it make sense to think that?" And then I asked the person who gave the answer to respond. This routine was a way of modeling talk about thinking. It also made thinking into a public and collaborative activity, wherein students would rehearse the sort of intellectual courage, intellectual honesty, and wise restraint that Polya considered to be essential to doing mathematics.

The Teacher as a Representation of What It Means to Know Mathematics

Simply by virtue of having had more education, the teacher represents the most expert knower of mathematics in the classroom and, in this role, has the potential to demonstrate the nature of expertise to those who seek to acquire it. Given my goal of teaching students a new way of knowing mathematics, I needed to demonstrate what it would look like for someone more expert than they to know mathematics in the way I wanted them to know it. The role I took in classroom discourse, therefore, was to follow and engage in mathematical arguments with students; this meant that I needed to know more than the answer or the rule for how to find it and I needed to do something other than explain to them why the rules worked: I needed to know how to prove it to them, in the mathematical sense, and I needed to be able to evaluate their proofs of their own mathematical assertions.[7]

For students to see what sort of knowing mathematics involves, the teacher must make explicit the knowledge she is using to carry on an argument

with them about the legitimacy or usefulness of a solution strategy. She needs to follow students' arguments as they wander around in various mathematical terrain and muster evidence as appropriate to support or challenge their assertions, and then support students as they attempt to do the same thing with one another's assertions. As the teacher moves around in mathematical territory in a flexible manner, she is modeling an approach to problem solving. She is demonstrating what it means "to know" what the mathematician Henry Pollak calls "cross-country" mathematics.

In contrast to walking on a well marked path, the cross-country terrain is jagged and uncertain; watching someone traverse it is a key to learning how to traverse it yourself. Pollak said of his teacher, Ed Begle:

> [As a student, I] had a very interesting time watching him struggle, inventing proofs and trying to think about the right way to do it. I learned a lot more mathematics that way than I might have if it had been a perfectly polished lecture; and I think already at that time I developed my feeling that I like cross-country mathematics. Mathematics, as we teach it, is too often like walking on a path that is carefully laid out through the woods; it never comes up against any cliffs or thickets; it is all nice and easy. (Albers & Alexanderson, 1985, p. 231)

If the teacher only demonstrates that she knows how to explain the rules, and whether or not students' answers are correct, the student will get an unfortunately limited picture of mathematical expertise, and it is unlikely that he or she would learn how to walk on any but the most well-marked paths.

Summary

These patterns of social interaction were designed to involve me, as the teacher, in three different kinds of teaching about what it means to know mathematics. Sometimes, I straightforwardly told students what kinds of activities were and were not appropriate. At other times, I modeled the roles that I wanted them to be able to take in relation to themselves and one another. And at other times, I did mathematics with them, just as a dance

instructor dances with a learner so that the learner will know what it feels like to be interacting with someone who knows how to do what he or she is trying to learn how to do.[8] Just as the dance instructor knows traditional forms of dance and demonstrates them, I demonstrated the conventions of mathematical discourse to my students, but we also reinvented them as we did mathematics together. Students also assumed the role of more experienced knowers in relation to one another as they became more competent in the interactional routines associated with mathematical discourse in the classroom. The lesson described below represents a moment in this process, when I would claim that the students had learned to regard themselves as a mathematical community of discourse, capable of ascertaining the legitimacy of any member's assertions using a mathematical form of argument.

An Episode of Teaching and Learning About Knowing Mathematics

Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of determining what is true. Students learn about how the truth of a mathematical assertion gets established in mathematical discourse as they zig-zag between their own observations and generalizations--their own proofs and refutations--revealing and testing their own definitions and assumptions as they go along.

In the lesson I will describe here, the mathematical content of the discourse was the operation of exponentiation: writing numbers as "powers" of other numbers and comparing their orders of magnitude. Exponentiation is a powerful mathematical tool, used both in the development of new theory and in designing applications of mathematics to the solution of scientific problems. The use of orders of magnitude for thinking about relationships among numbers

is attributed to Archimedes; his notions about comparing magnitudes by comparing orders of magnitude, or "powers," later evolved into the scientific notation and logarithms that are indispensable to applications of mathematics across the spectrum of scale, from microbiology to astronomy (Jacobs, 1970).

By comparing powers, we can understand measurement and counting (both in human experience and in science) in terms of hierarchies of scale, and we can appreciate something of the nature of mathematical knowledge in comparison to knowledge gained from physical experience. Familiar objects in our world span only six orders of magnitude: The smallest object we can see--the point of a fine needle, for example--is a few tenths of a millimeter in diameter, while at the other end of our spectrum of experience are objects smaller than a kilometer in any dimension; no animal has ever reached that size, and even massive buildings are less tall (Morrison, Morrison, & Eames, 1982). In terms of powers of ten, then, what we know directly is between 10^{-3} and 10^{3} meters.[9] With powerful telescopes and microscopes, scientists have been able to extend our experience over 42 orders of magnitude: The largest known dimension (the distance to the furthest star that has been perceived) is about 10^{25} meters, and the smallest particle of matter that has been identified (the quarks which make up the internal structure of protons) can be measured as 10^{-16} meters (Morrison, Morrison, & Eames, 1982).

Before Archimedes, the idea of actually counting something like the number of grains of sand in the universe (which is what Archimedes set out to figure out how to do), to say nothing of measuring galaxies and quarks, was inconceivable; Archimedes' tool made it seem doable. Several centuries later, another intellectual tool was invented which further enabled the use of orders of magnitude in mathematical problem solving. Descartes is attributed with the first use of the exponential form "x^2" to symbolize "x multiplied by

x." This was more than a mere shorthand, for it provided a basis on which new mathematical relationships could be conceived and represented and then operated upon with much simplified procedures. The concurrent development of logarithms by Napier led to entire new branches of mathematical theory and powerful scientific applications of quantitative tools (Kramer, 1970).

The power of exponents lies in the idea that you can compare numbers by comparing their orders of magnitude, thereby simplifying the mental operations involved in the comparison; one technical implication of that simplification is that you can multiply large numbers by adding exponents, and divide by subtracting them. To find out how many times wider a building that was 10^2 meters across was compared with a needle that was 10^{-3} meters across, for example, we would divide 10^2 by 10^{-3}, yielding 10^5. These kinds of calculations are puzzling, not only because they involve subtracting a negative from a positive number, but because they use addition and multiplication to combine quantities in ways that build one abstraction on top of another. It is easy enough to legitimate a sum by pushing two groups of objects together and counting the total or to legitimate a product by displaying a rectangular array. But how could anyone <u>prove</u> that it is legitimate to add and subtract exponents as a way of multiplying and dividing? Why is it true that 10^3 divided by 10^{-2} is 10^5? In mathematics we can prove that this is true, given certain assumptions about the meaning of exponents, by the process of mathematical argument: the process of observation and generalization that Lakatos and Polya describe.

The power of mathematics as a discipline distinct from science is that the truth thus generated can support assertions that go far beyond the realm of human experience. Once we know that it is true that numbers can be multiplied by adding their exponents, we have a cognitive technology for

understanding numbers like 10^{100}, 10^{1000} and beyond. Moreover, we can move to the realm of manipulating expressions like a^n and a^m, knowing for sure that their product is always a^{n+m}, given certain constraints on what is symbolized by a, m, and n.

The Teaching Agenda: Content and Process Intertwined

I wanted my students to learn not only that they could divide or multiply by subtracting or adding exponents and how to use the technology of exponents, but also that the warrant for doing so comes from mathematical argument and not from a teacher or a book. This meant that I needed to work on two teaching agendas simultaneously. One agenda was related to the goal of students' acquiring technical skills and knowledge in the discipline which could be called knowledge of mathematics, or mathematical content. The other agenda, of course, was working toward the goal of students' acquiring the skills and dispositions necessary to participate in disciplinary discourse, which could be called knowledge about mathematics, or mathematical practice (Ball, 1988; Schoenfeld 1985a; Wilson, 1988).

These two kinds of knowledge interact: In learning about discourse, students learn about what kind of knowledge they have when they know how to use exponents. At the same time, knowledge of tools, vocabulary, and symbols provides students with the "cognitive technologies" that enable them to make arguments of a substantially different sort than they would be able to make without them (Pea, 1987; Vygotsky, 1978). The interaction between learning to use the tools that have become a part of mathematical culture and inventing new mathematics by reasoning may be thought of as the process by which the individuals in the community of discourse that is the school class come to know mathematics. As such, it parallels the development of knowledge in the discipline.[10]

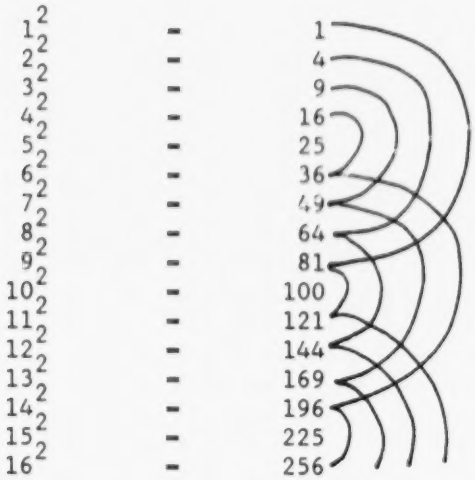
The Content of the Lesson as an Expression of the Nature of Mathematical Knowledge

The nature of mathematical knowing is such that you can find out something about the characteristics of unknown quantities by studying patterns in numbers that you can observe (Polya, 1954). This is the essence of the connection between learning about exponents and learning about mathematical knowledge. I started the unit on exponents by asking all of the students in the class to prepare their own tables of squares from 1^2 to 100^2 using calculators. I challenged them to find patterns in the tables, and they were actively engaged in doing so for at least three 45-minute periods.

During this activity, they invented a way of thinking about relationships among the numbers upon which I could build to take them into new mathematical territory: They developed the idea of talking about patterns in powers by focusing on the "last digits" of numbers, rather than thinking about all of the digits in a number at once. This was a tool, developed within the mathematical community of discourse in my classroom, that could be used to move away from the particular numbers that were represented in the list of square numbers, toward the characteristics of those numbers and the structure of relationships among them. The connection between learning about exponents as mathematical content and learning about the nature of mathematical knowledge by engaging in discourse about exponents is that, in mathematics, you can find out something about the character of unknown numbers by studying patterns in numbers that you know. For my class of fifth graders, looking for patterns in the "last digits" of powers was a way into knowing about what sort of predictive power can be derived from the analysis of quantitative order.

Looking at last digits is a mathematically appropriate cognitive tool; it reduces the amount of information you need to pay attention to in searching

for patterns. The first assertion that students in my class made about patterns in square numbers was that the last digits of the squares alternated between being odd and even, just as the base numbers did. Their most sophisticated conjecture about squares was that: The square of a multiple of 10 would always be a multiple of 10 and so end in a zero, the square of a number ending in 5 would also always end in a 5, and in between the zeroes and fives, the strings of last digits would always be symmetrical around both zero and 5. They used physical representations like this to work out this conjecture:



. . . and so on.

And they proved that the pattern would continue "forever" by arguing that numbers that ended in 1 or 9 would have to have squares that ended in 1, numbers that ended in 2 or 8 would always have squares that ended in 4, numbers that ended in 3 or 7 would always have squares that ended in 9, and numbers that ended in 4 or 6 would always have squares that ended in 6. Since the last digits of whole numbers are arranged in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, and this arrangement repeats itself over and over again, the square numbers would follow the doubly symmetrical pattern as long as there were base numbers to associate them with. It was at this point in the unit on exponents that the lesson I want to describe here occurred. I used the

students' idea of looking at what happens in last digits to elicit some more general conjectures about how to operate with exponents.

The lesson began when I wrote on the blackboard at the beginning of class:

What is the last digit in: 5^4?
6^4?
7^4?

and I challenged the class to tell me if they could prove that their conjectures about what these last digits would be were true without doing the full multiplications. Finding the "answers" to these three questions is a trivial activity, and it is made even more so because they can easily be obtained by using a simple calculator. But the mathematical content embedded in inventing the strategies that can be used to assert the answers <u>without doing the calculations</u> are mathematically significant and engage students in arguing about the key ideas behind how exponents work. The activity of developing such strategies engages students in clarifying the distinction between exponentiation and multiplication and leads to evidence that supports the mathematically legitimate "shortcut" of finding products by adding exponents. So although these are small questions, the problems entailed in finding ways to generate the answers to these questions can engage students in thinking about large ideas. The earlier lessons we had done on patterns in the last digits of squares laid the groundwork for students going beyond the trivial in their discussion of "fourth power" patterns.

After writing these questions on the board, I walked around the classroom, watching and listening to what the students were doing, and when everyone seemed to be engaged in the task, and most had given it some thought, I began a whole class discussion. That discussion had three parts: a clarification of terms, symbols, and definitions; a consideration of the

special properties of powers of 5 and 6; and speculations about powers of 7 that led to more general hypotheses about how exponents could be operated on to do arithmetic operations more efficiently. The discussion lasted for approximately half an hour and engaged almost every member of the class in generative mathematical activity.

Agreeing on What Exponents Mean

In the first part of the discussion, teacher and students were finding out what other members of the group understood about the operations indicated by exponents and coming to some agreement about common language to use in their conversation. Students started doing some "conscious guessing" about how exponents work as they were defining the terms of the argument. The first part of the class session focused on the fourth power of five. I began by saying, "Who is ready with a theory about this one?" pointing to 5^4 on the board.

Narinia asserted that, "It's going to be the same as the squares. First you take 5 and square it, and you get 25, and the last digit of that is 5, and you square that." Alianna then said, "Yeah. You square the 5 two times." Her tone of voice suggested agreement with Narinia, but what she said would not unambiguously describe the procedure that Narinia had followed. In previous classes, Alianna and other students had interpreted expressions written as "x^n" to mean "multiply x by n." We had had some discussion then about why exponential notation might lead to this confusion.

I wrote on the board:

> Alianna: Square five two times.
> Narinia: Square five and then square that.

I said, "I think we have a language problem here. Do those two directions mean the same thing?" Martha responded saying, "They might think you have to

square 5 and times it by 2." And I wrote "25 x 2" on the board again, saying, "Do you mean this?" At this point, Alianna came back into the conversation and said, "No, cause it's square it, again. Also, if you said 'Square 5 two times,' they might add it and get 50 [25 + 25]." So I asked, "How should you say it?"

Narinia reentered the discussion, clarifying what she had said earlier: "Square 5, and then square the answer to that." Carl also had his hand raised, and said, "Square the squared number of 5." And Gar piped in, "I was going to say what Narinia said." Martha then thought it appropriate to make a statement about how she interpreted the conversation: "I understood both of them. I did 5 times 5 and I did 5 times 5 again, but I didn't add them, I multiplied them."

Discussion[11]

There are several mathematically legitimate ways to figure out the last digit in 5^4. One could multiply 5 x 5 and then multiply the answer to that by 5, and multiply the answer to that by 5: 5 x 5, then 5 x 25, then 5 x 125, to get 625. What the students were trying to figure out--the problem they were solving--was whether there was a more efficient way to do it that was also mathematically defensible. As they argued about this, they clarified their language and use of symbols.

Students and teacher need to be able to have a conversation using terms that are functional, not only for communication but for reasoning. As a representative of mathematical culture outside of the classroom, the teacher brings conventional mathematical tools (including language and symbols) into the discussion and negotiates their meaning with students to add to the tools they are able to use to enhance their thinking. In the first part of the

class this entailed the teacher checking in with those students who had in past discussions interpreted "the power of 2" to mean "multiply by 2." Although exponentiation entails repeated multiplication, just as multiplication entails repeated addition, it is a qualitatively different operation from multiplication and it works by different rules (Asimov, 1961). Unless this difference between multiplying by 2 and the power of 2 was cleared up, these students would be unable to productively participate in arguments about how the fourth power of a number might be related to the second power and go on to assert appropriate rules for working with powers.

Conscious Guessing and Proof That Zig-Zags Between Induction and Deduction

After this discussion of how we would talk about the operations indicated by exponents, Sam asserted, about the last digit in 5^4: "It has to end in a 5." I invited everyone in the class to consider the validity of Sam's decisive assertion and to see if they could explain why he seemed to be so sure. The question I was asking was: How does he know that is true? Harriet said, "Well, anything multiplied by 5 has to end in a 5 or a zero," and Theresa quickly added, "but it has to be a 5 because when you multiply 5 times 5 you get a 5 [for a last digit]." Martha observed, "You times the square number, you square it again and you get 625." And Carl responded, moving to the level of a mathematical generalization, "You don't have to do that. It's easy, the last digit is always going to be 5 because you are always multiplying last digits of 5, and 5 times 5 ends in a 5." Carl went beyond the question about the last digit in 5^4 and gave both a conjecture and a proof about what must be true of the last digit of all of the powers of 5.[12]

At this point, some other students, who had been working at finding higher and higher powers of 5 with their calculators, made the conjecture that all powers of 5 would end in "25" for their last two digits, and they made some arguments for why that might "always" be true, no matter how high the power. In doing so, they also zig-zagged between observations and generalizations. Interestingly, their deductive arguments relied on the framework of the conventional multiplication algorithm, even though they had been using the calculator to develop their hypothesis inductively: They talked about what would "have to happen" in the "ones" and the "tens" columns every time you multiplied and carried in successive multiplications of a number that ended in "25" by five.

We had a short discussion of the powers of 6, which seemed to be of little interest because "they worked just like fives" (i.e., everytime you multiply a number that ends in 6 by another number that ends in 6, the result will end in 6). There was a conjecture that this similarity in last digits might be true for the powers of every number from 1 to 9, but someone quickly refuted it with a counterexample: 7^2 is 49; the power does not end in the same digit as the base number. Some students were talking among themselves about the fact that the last two digits in the powers of 6 were not always "36," which they had expected, based on an analogy to 5 where the last two digits were always 25.

Discussion

In this part of the lesson, students had begun to make assertions that were based on their inductive observations of patterns and to move back and forth between these observations and deductive arguments about why the patterns would continue, even beyond the numbers they had tested. They ranged

beyond speculating about the fourth power, which was the focus of the initial problems, and made conjectures about all of the powers of 5 and 6. They began by stating reasons why anyone might "know for sure" that the last digit of a power of 5 would always be 5, and then went on to explore the boundaries of the theory that "If the last digit is always n when you raise n to some power, the last two digits will always be n^2." This theory worked for n = 5; they proved that the last two digits of any power of 5 would be "25" (or 5^2). Because the powers of 6 always ended in a 6, they wondered if the two-digit conjecture would also work for n = 6. Two students asserted that (i.e., stating their idea as collaboration: "Me and Hudson think that . . .") the powers of 6 should end in 36, but someone else quickly found a counter-example.

In the course of this discussion, students shifted around from talking about "What I did" to figure out the problem to "What you do," distancing themselves from the procedures they were evaluating and making their assertions more general. In this process, their beliefs became justified and they moved into the realm of mathematical truths that were legitimated by the community of discourse.

There is evidence in this segment of the lesson that students appreciate that mathematics is about finding efficient strategies to solve problems; in addition to the student-initiated search for two-digit patterns, this showed up when Carl made a point of saying that "You don't have to figure out that 5^4 is 625 in order to know that its last digit must be 5," and then went on to support his argument with mathematical evidence. He was calling the class's attention to what doing mathematics means and giving a particular example of what is entailed in doing it. It is not that multiplying 5 x 5 x 5 x 5 does not prove that the last digit in 5^4 is 5, but this proof is

particular, not general; it is not as elegant as one that focuses only on what must be true about the last digit that results from each step in the operation.

When the Answer Is Not the End of the Discussion

When I asked if anyone had any ideas about 7^4, there was quick agreement that it should be 1 and I asked for proof. Gar said: "7 times 7 is 49. And 9 times 9 is 81. So take the 1." Embedded in this argument is the assumption that $7^4 = 7^2 \times 7^2$ and that all you need to attend to is the last digit all the way through the procedure. Gar multiplied 9 x 9 to arrive at 1; he did not multiply 49 x 49, and then try to figure out the answer to that. Versions of his argument were given and elaborated by other students. There was a repetition in much of what they said of the language that had been used in discussing 5^4, but the same terms and sentence structures were being used by different students. By this point, nearly everyone had had something to say.

Then I asked, to further elicit their conjectures about how exponents work: "What about 7 to the fifth power?" Several students raised their hands after a few moments, and the ones that were called on said in quick succession:

Arthur: I think it's going to be a 1 again.
Sarah: I think it's 9.
Soo Wo: I think it's going to be 7.
Sam: It <u>is</u> a 7.

I wrote on the board: $7^5 =$ 1?
9?
7?

and said: "You must have a proof in mind, Sam, to be so sure," and then I asked, "Arthur, why do you think it's 1?"

The following discussion ensued as the students attempted to resolve the problem of having more than one conjecture about what the last digit in 7 to the fifth power might be. It was a zig-zag between proofs that the last digit must be 7 and refutations of Arthur's and Sarah's alternative conjectures. The discussion ranged between observations of particular answers and generalizations about how exponents--and numbers more generally--work. Students examined their own assumptions and those of their classmates. I assumed the role of manager of the discussion and sometimes participated in the argument, refuting a student's assertion.

Teacher: Arthur, why do you think it's 1?

Arthur: Because 7^4 ends in 1, then it's times 1 again.

Gar: The answer to 7^4 is 2401. You multiply that by 7 to get the answer, so it's 7 x 1.

Teacher: Why 9, Sarah?

Theresa: I think Sarah thought the number should be 49.

Gar: Maybe they think it goes 9, 1, 9, 1, 9, 1.

Molly: I know it's 7, 'cause 7 . . .

Abdul: Because 7^4 ends in 1, so if you times it by 7, it'll end in 7.

Martha: I think it's 7. No, I think it's 8.

Sam: I don't think it's 8 because, it's odd number times odd number and that's always an odd number.

Carl: It's 7 because it's like saying 49 x 49 x 7.

Arthur: I still think it's 1 because you do 7 x 7 to get 49 and then for 7^4 you do 49 x 49 and for 7^5, I think you'll do 7^4 times itself and that will end in one.

Teacher: What's 49^2?

Soo Wo: 2401.

Teacher: Arthur's theory is that 7^5 should be 2401 x 2401 and since there's a 1 here and a 1 here . . .

Soo Wo: It's 2401 x 7.

Gar: I have a proof that it won't be a 9. It can't be 9, 1, 9, 1, because 7^3 ends in a 3.

Martha: I think it goes 1, 7, 9, 1, 7, 9, 1, 7, 9.

Teacher: What about 7^3 ending in 3? The last number ends in . . . 9 x 7 is 63.

Martha: Oh . . .

Carl: Abdul's thing isn't wrong, 'cause it works. He said times the last digit by 7 and the last digit is 9, so the last one will be 3. It's 1, 7, 9, 3, 1, 7, 9, 3.

Arthur: I want to revise my thinking. It would be 7 x 7 x 7 x 7 x 7. I was thinking it would be 7 x 7 x 7 x 7 x 7 x 7 x 7 x 7.

Once Arthur revised his idea that the last digit should be "1" there were no further disagreements in the class with the conclusion that it should be "7." There was little time left, and I used it to extend into a different domain the hypotheses students had been developing about how exponents work.

Teacher: What power's <u>that</u>? (i.e. 7 x 7 x 7 x 7 x 7 x 7 x 7 x 7)

Arthur: 8th.

Teacher: That's 7^4 squared.

(On board: $7^8 = 7^4 \times 7^4$)

Teacher: What's 7^{16}, Arthur ?

Arthur: It's going to be $7^8 \times 7^8$.

Julio: I think $7^{16} \times 7^{16}$ is going to be 7^{32}. It just doubles.

Soo Wo: Since 7^3 is 343, I think 7^6 would be $7^3 \times 7^3$, which would be 343 x 343.

(On board: $7^4 = 7^2 \times 7^2$)

Teacher: Time is up. We have to stop. We'll continue on Monday.

At this point, some students were verging on declaring the second law of exponents: $(n^a)(n^b) = n^{a+b}$, which they would articulate more fully, and prove the legitimacy of, in the next few classes. They were also beginning to develop a modular arithmetic of "last digits" to go with different base numbers, leading them into further generalizations about the properties of exponents.

Discussion

By the end of the lesson, 14 of the 18 students present in the class had had something mathematically substantial to say about exponents: an interpretation of language or symbols, an assertion about a pattern, a proof that a pattern would continue beyond the observed data, or an interpretation of another student's assertion. Each of these kinds of contributions is both an expression of what the student knows and a rehearsal of what the student believes to be an appropriate contribution to a school lesson in mathematics.

This final segment of the lesson represented a shift from the questions that were posed at the beginning of class to a new kind of question. It was an extension of the domain of discourse from patterns that would apply to numbers raised to the fourth power, to strategies for figuring out what would be the last digit in a number raised to the fifth power. In the case of 5 and 6 as a base number, the question was trivial, but in the case of 7, it was not. The new question provides a forum in which assumptions that might have been made about how exponents work in a special domain could be tested on a new population with different characteristics. From the teacher's point of view, this shift has at least two functions. It extends the content of the lesson into new territory, and it serves to assess students' "conceptual competence" (Greeno, Riley, & Gelman, 1984) by assessing whether they could bring a conceptual structure developed in a more familiar domain to bear on one that was less familiar and more complex.

I purposely did not ratify any of the students' assertions about the answer to 7^5, nor their arguments for their various positions. When Arthur says, "I want to revise my thinking," he is using a phrase that he and the rest of the class have been taught to use and encouraged to use when they want to change their minds about an assertion made earlier in the discussion. It

carries a different message than saying "My answer was wrong, and now I know the right one." When a student is in charge of revising his or her own thinking, and expected to do so publicly, the authority for determining what is valid knowledge is shifted from the teacher to the student and the community in which the revision is asserted.

Arthur maintained his intellectual courage through the course of this discussion, not changing his belief without revising the assumptions on which it was built. He was not inclined to revise by the repeated assertions of other students that the last digit was 7 because he had made the guess that 7^5 would be obtained by multiplying 7^4 by 7^4, perhaps because we had established that $7^4 = 7^2 \times 7^2$. He was searching for a mathematical law, rather than just looking for the answer, and his conjecture was that the next highest power would be obtained by squaring the power below it. He revised his conjecture to assert that squaring a power would result in a power that was double the one you started with, so that 7^{16} could be obtained from the product of 7^8 and 7^8, 7^{32} could be obtained from the product of 7^{16} and 7^{16}, and so on. Beginning with a guess and exploring it with courage and modesty, he arrived at a bit of mathematical truth. What he was learning was both the laws of exponents and how to justify that they work within the domain of mathematics.

Gar also exhibited intellectual modesty in his argument against Sarah's assertion that 7^4 would end in a 9. He worked at figuring out why she might think that and then explained the logical contradiction in her assumption. He did not tell Sarah she was wrong. He left that judgment to her. Sarah never does get to tell the class why she thought the last digit should be 9, and there is no verbal record of whether she revised her thinking. (One would need to explore Sarah's thinking about exponents on some later occasion to

find out more about whether her ideas about how they worked were congruent with mathematical conventions.) The same could be said of the interaction between Martha and Sam over whether the last digit could be 8.

The role that I took as the teacher, in relation to Arthur's assertion that the last digit in 7^5 should be 1, was to ask him to explain, to monitor the tone of other students who wanted to disagree, to assert what some of Arthur's assumptions might have been, and to ask Arthur to articulate the revision that he had made. Throughout this part of the class, the teacher acted to support the idea that "Arthur's thinking" should be everyone's concern. The lesson was not just getting Arthur to rethink his assumptions, but helping everyone to see why those assumptions had led him to the conclusion that 7^5 would end in a 1. Both students and teacher have a different relation to the subject matter in this kind of discourse than they would in a conventional "knowledge telling" exchange.

Arguments for the conclusion that the last digit in 7^5 should be 7 were given by other students repeatedly throughout the discussion. These arguments could be interpreted as "knowledge telling" by students who wanted the teacher to know what they were thinking, but they had the dual purpose of attempting to convince other members of the class that this conclusion was valid. Since they were asserted interactively with other assertions, there is some evidence that they may have been intended to have this argumentative purpose. On many occasions when such discussions occurred, one student might give several different arguments for why his or her conclusion was valid, and sometimes students would even ask to come up to the board to illustrate an argument for the class.

Worth noting in this part of the discussion is how often (15 times in about 5 minutes) and how many (8 out of 20) students say "I think" or "I don't

think" and then make an assertion. An analysis of their assertions shows that they are not mere expressions of opinion; they recount their own reasoning processes and analyze those of others. They use "I think" to mean "I have figured out that," and their assertions of what they have figured out are regularly followed by arguments for why their strategies seem valid. They are indicating authorship of the ideas that they assert, and they are also indicating that thinking is something that a student both does and talks about in a mathematics class. Making assertions in this form is an expression of what they believe about roles and responsibilities in relation to mathematical knowledge and where they put themselves in relation to the establishment of valid arguments in the discipline.

This use of language also serves to signify where the student is in the mathematical journey from a conscious guess to a proven theorem. Students who are sure that they can support their assertions say "It is" or "It has to be" or "I know." Saying "I think" rather than "It is" protects the student from associating his or her sense of self with an assertion that is later revised because it has been proven wrong.[13] This level of engagement with mathematical ideas may be similar to the process of exploration or "free writing" that occurs before one constructs a final draft (Barnes, 1976). In using this terminology from writing to describe classroom discourse more generally, Barnes asserts,

> The difference between exploratory and final draft is essentially a distinction between different ways in which speech can function in the rehearsing of knowledge. In exploratory talk and writing, the learner himself takes responsibility for the adequacy of his thinking; final-draft talk and writing looks toward external criteria and distant unknown audiences. (pp. 113-114)

It requires courage and modesty to expose one's exploratory thinking to others in the hopes that by engaging in the exchange of ideas in classroom discourse, one might end up with better ideas in the end.

This final segment of the lesson most clearly exhibits what the students had learned about the nature of mathematical knowledge and about their responsibilities as participants in the activity of learning mathematics. When they disagreed with their classmates, the disagreements took the form of presenting evidence that was intended to prove that what had been asserted could not be true. The evidence appropriate to mathematical discourse is an argument that demonstrates that an assertion will lead to a contradiction, given the domain in which the discourse is conducted, or a counterexample. Students were often mustering such evidence in an attempt to defend their own assertions and refute the assertions of others at the same time. Their choice of refutation as a form of disagreement with their peers instead of put downs or silence--forms more common to discourse among fifth graders--is a significant indication of what they believe about how truth is established in mathematics: it is not established by the teacher, or another student, saying that an answer is right or wrong but by mustering the evidence to support or disprove an assertion, be it one's own or that of a classmate.

If That Is the End of the Story, Then What Is the Beginning? Some Nonmathematical Ways of Knowing Mathematics in School

In order to support the argument that these students had learned something about mathematics, I need to substantiate the claim that they were doing something novel in this discussion or that they were not doing something that they had been doing before and that students of their age in school do regularly. In this section of the paper I will describe several patterns that characterize students' approaches to learning and knowing mathematics in the classroom before they have learned to participate as active members of a community of discourse about mathematics. Although they are fitting as constructs based on students' experiences with learning mathematics in school,

these approaches are in conflict with those that Lakatos, Polya, and others consider to be appropriate to doing mathematics. That conflict gets worked out in the classroom as my students and I negotiate through the year about what our respective roles and responsibilities will be. I take the patterns in student behavior that I will describe here as a starting point for instruction about the nature of mathematical knowing. There are fewer of these behaviors in my classes toward the end of the year than at the beginning.

1. Turning to the Teacher or Some Other Reliable Authority for Ratification

Perhaps the question I am asked most often by observers of my teaching is, "Don't you ever tell them whether their answers are right or wrong?" Students (and many other observers of teaching and learning) expect that this is part of the teacher's role, and they are disconcerted when the teacher does not comply. They ask, "How can I tell whether I am doing well, if you will not tell me whether the answer is right?" or "How do I know if I know my math if you won't check it for me?" Once students figure out that their teacher will not be cajoled into doing the work of establishing the validity of their results, many students look for verification from peers whom they know from previous classes usually get the right answer. Even if they do not ask these students for help directly, they look to them in discussions and feel more confident if their own assertions match those of the students they identify as "smart in math." On occasion, when I have asked a student why he or she revised an assertion at the beginning of the school year, I would be told straightforwardly, "Because that's what Tommy said, and he's usually right."

2. Treating Rules, Formulas, and Facts as if They Were Arguments

Particularly among students who have been successful at school mathematics because they are good at memorizing and following rules, there is a tendency to use rules as reasons for action, as if they could stand in for explanations. They answer questions about why it seems appropriate to use a particular rule in a given problem with a repetition of the rule or perhaps a reference to the person who taught it to them. These students are generally resistant to and impatient with the representation of numerical calculations in another medium, and they are often unable or disinclined to express the relationship between arithmetic operations and the actions on quantities they are meant to indicate. They are disconcerted when their ability to use a conventional algorithm correctly goes relatively unrewarded by the teacher while students who (from their perspective) have arrived at the "wrong" answer are praised for the questions they have raised or the way they have represented the problem. Once they have arrived at an answer by correctly following the rules, not only do they not consider it their responsibility to listen to or argue with other students' answers, but also, they act in ways that are disruptive to carrying on a discussion.

3. Keeping Thinking Implicit or Private

One common sort of student behavior that does not serve well in the creation of mathematical discourse in the classroom is silence. In particular, silence as a way of expressing disagreement with an assertion is in conflict with the notion that knowing mathematics involves arguing, defending, challenging, and proving one's own ideas and those of others. It is not impossible that these activities could take the form of written rather than verbal communication (as they often do in communication among mathematicians), and that approach would be plausible in a situation where students

have the tools, the time, and the skills to express their thinking in writing. But these conditions do not obtain in most elementary and many secondary classrooms.

Often students who have not had much experience with the process of discussing mathematical ideas will answer questions about how they figured something out with phrases like, "I just know" or "I just thought it" or "I don't know how I figured it out." These responses are often delivered in a tone of voice that further suggests, "And besides, it's none of your business or anyone else's how I got my answer." Sometimes these responses are delivered in a tone that indicates that the student does not have the words to tell anyone what mental processes led to a particular conclusion. At other times the student's tone suggests that he or she lacks the courage to expose the thinking behind an asserted answer to teacher and class for comment or is uncomfortable with having the class pay attention to that thinking. On occasion, this response can be taken as an indication that the student saw the answer on the paper of a classmate at the next table and knows telling that to the teacher would not be well received, either by the teacher or by the student with the answer.

4. Disagreeing by Exerting Physical or Political Power Over Peers

If students come up with different answers to a problem, it is not unusual for them to try to shout down the opposition or more indirectly to intimidate someone who disagrees. Students do not come to fifth grade assuming that it is inappropriate to characterize one of their fellow classmates as "dumb" or "stupid" when that person has made what they consider to be an obvious error. Even when they have learned that this sort of talk will not be tolerated in large group discussions, they address their peers that way in small group problem-solving activities.

A more civilized variant on this theme and one related to other patterns described above is students' suggesting in a large- or small-group discussion that we resolve a difference among conjectured answers by voting, without having to listen to why anyone thinks one or another answer is more valid. They argue that voting would resolve the disagreement without exposing the incorrect assumptions or procedures that led to the divergence in the first place. From the perspective of the student who often gets the correct answer, this is sometimes offered as a ploy to push the class to "get on with it" in a way that also gets them some rewards because they can rely on less secure students to vote for their answers.

5. Stubbornness and Face-Saving Behavior

Some students will stick with an assertion even after several arguments have been made against it, saying that the answer is correct because it was obtained by "my way of doing it." They do not distinguish between constructing mathematically legitimate reasons and the act of an individual thinking something up. They act as if they believe that admitting that there is something wrong with their reasoning is an admission that there is something wrong with them. They refuse to expose their assumptions or they engage in very clever casuistry to explain why their assumptions were valid. They turn the idea that multiple solutions to a problem are possible into the relativistic notion that every solution to a problem should be accepted just because someone came up with it.

These actions on the part of students are only inappropriate if one takes the view that knowing mathematics means engaging in mathematical discourse. Just as we assume that students' constructs of mathematical ideas make sense within the domain of their experience, we need to assume that these social behaviors make sense as an expression of what they think it means to do

and to know mathematics in school. If the lesson on exponents had been taught without regard for students learning about the nature of mathematical knowledge, these behaviors would not have been out of place.

Conclusion

As I went about trying to teach my students to give up this familiar set of behaviors for what Polya calls "the moral qualities of the scientist," I assumed that they would not learn a different way of thinking about what it means to know mathematics simply by being told what to do, anymore than one learns how to dance by being told what to do. I assumed that changing students' ideas about what it means to know and do mathematics was a matter of immersing them in a social situation that worked according to different rules than those that ordinarily pertain in classrooms, and then respectfully challenging their assumptions about what knowing mathematics entails. Like teaching someone to dance, it required some telling, some showing, and some doing it with them, to say nothing of regular rehearsals.

I do not claim that my students' having learned to do mathematics in a way that is congruent with disciplinary discourse is entirely attributable to the social structure I created in my classroom. Students come to any class with varying degrees of expertise and experience in having these kinds of mathematical discussions, some having none and others having a great deal. There are other teachers in the school where I work who also have tried to engage their students in mathematical discourse over the past four years.[14] There is a mathematics coordinator in the district who understands what mathematical discourse is about and believes it to be an appropriate teaching method. My students are taught by several other teachers in different subject matter areas who, in terms of the discourse structures of their disciplines, also expect students to be the authors of ideas. And the students also have

parents, of course, at least one of whom is a professional mathematician. But they do act differently toward mathematical knowing at the end of the year than they do at the beginning in my class, and informal questioning suggests that they even carry some expectations about how one does mathematics from my classroom into other settings.

My arguments about what is entailed in teaching students about the nature of mathematical knowledge draw on work in the history and philosophy of mathematics and from research on teaching and learning in the tradition of social psychology. My organizing ideas have been the "humility and courage" that Lakatos and Polya take to be essential to doing mathematics. I have treated these as social virtues, and I have explored whether and how they can be deliberately taught, nurtured, and acquired in a school mathematics class. The evidence that they <u>can</u> answers my initial research question, but raises deeper questions about what students are actually learning and whether mathematical discourse is what they should be learning in school.

An exploration of these questions would be well served by considering how the activity I have portrayed as "mathematical discourse" looks from other <u>theoretical</u> perspectives, namely: the psychological analysis of individual cognition and the anthropological analysis of the knowledge involved in solving nonschool, nondisciplinary problems. From these perspectives, we might ask: What is to be gained from students learning to do mathematics with humility and courage, from their learning to generate knowledge in argument with one another in a manner similar to the way it is generated in the discipline? Are students' individual conceptual structures enhanced by this sort of activity? Is the mathematical knowledge acquired in the context of mathematical discourse more usable for the solution of the sort of unspecialized and messy practical problems that students will face once they

leave school? How is learning about mathematics as a social construction in a community of discourse different from learning to do mathematical reasoning by interacting with a computer? or a tutor? How is learning mathematics in the elementary school classroom distinguished from learning it from an apprenticeship in a community of working mathematicians?

Further questions follow if we ask whether mathematical discourse should occur in school from various practical perspectives: What do you need to know to teach this way? How is that knowledge acquired? How much more time and energy does this kind of teaching demand of teachers? What other resources besides skill and knowledge might teachers need to do this kind of teaching? How can the results of such teaching be portrayed to the public in such a way as to communicate that teachers are teaching and students are learning what they need to learn? Knowing what mathematical discourse might look like in the fifth grade, and establishing that it is possible for it to happen there, is only the beginning of what looks to be an interesting but difficult exploration.

Footnotes

[1]The common definition of axiom is "self-evident truth" and for a long time, axioms were regarded as such in mathematics. It was not until logicians found that the same set of self-evident truths could lead to contradictory conclusions, rather than certainty, that this view came into question in the discipline. (See Kline, 1980; Lakatos, 1976, p. 56.)

[2]They are also different from the classical logicist or Platonic view of mathematics as the discovery of ultimate truth that can not be refuted because of its logical foundations. (See Hadamard, 1947; Hardy, 1940; Hoffman, 1987.)

[3]The study reported here is part of a larger study, Teaching Mathematics for Understanding and Understanding Mathematics Teaching, funded by the Spencer Foundation through the National Academy of Education. The data are drawn from my own classroom teaching. I have taught fourth- and fifth-grade mathematics during the past six years, collecting data on both teaching and learning during three of those years. These data included audiotapes of lessons for six months, videotapes of two units, records of speech and visual communication kept by an observer at least three times a week over three years, notebooks in which students did their daily work including the writing and drawing they do to represent their thinking, and students' homework papers. I have also kept my own detailed field notes on lessons, including descriptions of how lessons and units were planned and implemented and initial analyses of the planning process itself, the lessons as they were taught, and students' work. I have further analyzed a portion of these data using triangulation among different data sources and constant comparison within and between lessons.

[4]The application of such ethnomethodological and sociolinguistic frameworks to the study of mathematics teaching and learning has also been advocated by Bauersfeld (1979) and Steiner (1987). Both argue that mathematics learning is a process of social as well as individual construction and that patterns of interaction are powerful in shaping both students' beliefs about what doing mathematics means and the sorts of activity they are inclined to engage in during a mathematics lesson.

[5]Being both the analyzer of the teacher's role and the incumbent, my use of first and third person in this paper is admittedly schizophrenic.

[6]An example of how this hypothesis testing works in relation to a more conventional topic in the upper elementary curriculum--comparing the relative magnitude of two decimal numbers--is described in Lampert (in press-a).

[7]I also initiated them into the use of conventional mathematical tools and language. For a more detailed explication of this aspect of the teacher's role, see Lampert (1986a, 1986b, in press-a).

[8]These roles are similar in many ways to those described by Palincsar and Brown (1984) in their presentation of "reciprocal teaching." Following the Vygotskian notion of "scaffolding" the teacher's participation fades somewhat as students become more expert in the pattern of discourse that is being taught. A difference between teaching mathematics and teaching reading

comprehension that seems worth exploring has to do with the interaction between learning content and learning discourse structures. At issue is the question of how much and what sort of mathematics a teacher would need to know to teach content using this method (Ball, 1988). Work currently in progress on the explicit teaching of strategies to enable students to become more independent learners of mathematics has focused on students learning problem-solving strategies rather than on teaching and learning what is conventionally thought of as mathematical "content" (Brown, 1988; Resnick, in press). Some studies of group problem solving have addressed learning in school mathematics as a function of dialogue in small group or paired interactions (Noddings, 1985; Schoenfeld, 1985b), and Schoenfeld (in press) has explored the teacher's role as a more expert problem solver who models "ways of knowing" in college mathematics classes.

[9]The symbol 10^{-3} means one one-thousandth, or 1/1000, which is $1/10^{3}$. Negative powers of 10 extend the place values system we use to build whole numbers into decimal fractions.

[10]See Cassirer (1957) for a general explication of how symbolic tools function in the development of human knowledge.

[11]After each part of the lesson is described, I will attempt to explain some aspects of my teaching, but these post hoc representations of my thinking are not meant to suggest what I may have been thinking at the time. Reflective justification is not the same thing as making reasoned choices about what to do within the act of teaching lessons. Actions may be warranted after the fact by arguments from mathematics, psychology, or pedagogy, but these warrants are not the same as reasons for doing one thing or another. In practice, teachers often are choosing between two or more courses of action, each of which is sufficiently warranted, but any one of which might be in logical conflict with another. (See Ball, 1988; Buchmann, 1988; Lampert, 1985.)

[12]The assumed domain for exponents is whole positive numbers at this point in the discourse.

[13]A recent paper by Cobb, Yackel, & Wood (in press) examines the teacher's role in shaping the way students interpret their feelings about participating in mathematical discourse in the classroom, with a focus on the students' developing capacity to distance themselves from their own activity to attempt to understand alternative points of view and resolve conflicts between incompatible solutions.

[14]I have taught other teachers to use the strategies I have described in this paper, and they are using them at different grade levels throughout elementary and middle school (Lampert, in press-b). Studies of their teaching practice are underway (e.g., Weinberg & Parker, 1988).

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